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A new approach to study the dynamics of the modified Newton's method to multiple roots

by

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Abstract

A new approach to study the dynamics of the Modified Newton's method due to Schröder is presented. This is a very simple but general approach that allows the study of the dynamics of methods to solve nonlinear equations, particularly when these have two roots with different multiplicity. Then, using the classical procedure to study the dynamics of iterative methods in the Riemann sphere, the stability of the fixed points and the parameter space associated with the critical point obtained are studied. Finally, dynamical planes and basins of attraction that confirm the results are shown.

Key Words: Nonlinear equations, modified Newton's method, dynamics, multiple roots.

2010 Mathematics Subject Classification: Primary 65H05.

1 Introduction

In 1870 Schröder [16] use the function $\sqrt[m]{f(x)}$ in the iteration equation of Newton's method to obtain a new method what he called Modified Newton's method.

$$z_{r+1} = z_r - m\frac{f(z_r)}{f'(z_r)}; \quad r = 0, 1, 2, \cdots \tag{1.1}$$

This method is of second order for the approximate calculation of a root of f with multiplicity m. Recently in [6], the author presented two geometrical constructions of said iteration equation. In particular, it can be obtained from the curve

$$y = f(x_r)\left(1 + \frac{f'(x_r)(x - x_r)}{mf(x_r)}\right)^m \tag{1.2}$$

If m is a natural number, (1.2) is a polynomial of degree m.

A classical way to analyze the behavior of iteration equations like (1.1) is through the study of the dynamics of these equations for classes of polynomials in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Some representative works on these studies can be observed in [1], [3]-[5], [7]-[11], [13]-[15] and [17]-[19]. The definitions of orbit, fixed points, critical points, stability, periodic orbits among others can be consulted in [2], [12] and the references mentioned above. In [18] Weifeng Yang presented a study of the standard, multiple and Modified Newtons method for polynomial $P(z)$ where the symmetries of Julia sets is investigated. In [9] and [11] the authors study the dynamics of damped Newtons method applied to different polynomial equations. In this paper, we study the dynamics of Modified Newton's method (1.1) for the class of polynomials with two roots with different multiplicities using a parameter (K) that depends on the relationship between them.

2 Dynamics of the Modified Newton's method

2.1 Conjugacy classes

Throughout the remainder of this paper we study the dynamics of the rational map R arising from the method (1.1)

$$R_f = z - m\frac{f(z)}{f'(z)} \tag{2.1}$$

applied to a generic polynomial $p(z) = (z-a)^m(z-b)^n, a \neq b$ with $m = Kn$ and real part of m and n greater than zero. First, it is necessary to establish the conjugacy class and the analytical expressions for the fixed and critical points of this method in terms of the parameter K.

Definition 1. *[2]. Let f and g be two maps from the Riemann sphere into itself. An analytic conjugacy between f and g is an analytic diffeomorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$.*

For R_f in (2.1), the following result holds.

Theorem 1. *(The Scaling Theorem). Let $f(z)$ be an analytical function on the Riemann sphere, and let $T(z) = \alpha z + \beta$, $\alpha \neq 0$, be an affine map. If $g(z) = f \circ T(z)$, then $T \circ R_g \circ T^{-1} = R_f(z)$. That is, R_f is analytically conjugate to R_g by T.*

Proof. we have

$$R_g(T^{-1}(z)) = T^{-1}(z) - m\frac{g\left(T^{-1}(z)\right)}{g'\left(T^{-1}(z)\right)}$$

Since $\alpha T^{-1}(z) + \beta = z$, $g \circ T^{-1}(z) = f(z)$ and $\left(g \circ T^{-1}\right)'(z) = \frac{1}{\alpha}g'\left(T^{-1}(z)\right)$, we get $g'\left(T^{-1}(z)\right) = \alpha\left(g \circ T^{-1}\right)'(z) = \alpha f'(z)$. We therefore have

$$\begin{aligned} T \circ R_g \circ T^{-1}(z) &= T\left(R_g(T^{-1}(z))\right) = \alpha R_g\left(T^{-1}(z)\right) + \beta \\ &= \alpha T^{-1}(z) - m\frac{\alpha g\left(T^{-1}(z)\right)}{g'\left(T^{-1}(z)\right)} + \beta = z - m\frac{f(z)}{f'(z)} = R_f(z) \end{aligned}$$

□

Definition 2. *[10]. We say that a one-point iterative root-finding algorithm $p \to T_p$ has a universal Julia set (for polynomials of degree d) if there exists a rational map S such that for every degree d polynomial p, $J(T_p)$ is conjugate by a Möbius transformation to $J(S)$*

Now, we establishes a universal Julia set for $p(z) = (z-a)^m(z-b)^n, a \neq b$ and $m = Kn$ for the equation (1.1).

Theorem 2. *For a rational map $R_p(z)$ arising from the method (1.1) applied to $p(z) = (z-a)^m(z-b)^n, a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{z^2}{(1-K)z + K} \tag{2.2}$$

where $m = Kn$

Proof. Let $p(z) = (z-a)^m(z-b)^n, a \neq b$ with $m = Kn$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse given by $M^{-1}(u) = \frac{bu-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{bu-a}{u-1}\right) = \frac{u^2}{(1-K)u+K}$$

□

We observe that parameters a and b have been obviated in $S(z)$, as an effect of the Scaling Theorem that is verified by this family. The use of $m = Kn$ also allowed the elimination of parameters m and n. Thus, the function of iteration to be studied only depends on the K parameter. This allows us to study many elements of the class of polynomials $p(z) = (z-a)^m(z-b)^n$ simultaneously; for example when $m = 3$ and $n = \frac{3}{2}$ has the same dynamics as for $m = 2$ and $n = 1$; since in both cases $K = 2$. It is obvious that when $m = n$, the dynamic behavior for $p(z) = (z-a)^m(z-b)^n$ is the same as for the case of $p_1(z) = (z-a)(z-b)$, in which case $S(z) = z^2$.

2.2 Stability of the fixed points

The fixed points of $S(z)$ are the roots of the equation $S(z) = z$, that is, $z = 0$, $z = \infty$ and $z = 1$. Now, we calculate the first derivative of $S(z)$ given in (2.2),

$$S'(z) = \frac{z\left[(1-K)z + 2K\right]}{\left[(1-K)z + K\right]^2} \tag{2.3}$$

Is obvious from (2.3) that $z = 0$ and $z = \infty$ are superatractive fixed points. The stability of $z = 1$ changes depending on the values of the parameter K.

The operator $S'(z)$ in $z = 1$ gives

$$|S'(1)| = |K+1|$$

In the following result we present the stability of the fixed point $z = 1$.

Theorem 3. *The strange fixed point $z = 1$ satisfies the following statements:*

1. *If $|K+1| < 1$, then $z = 1$ is an attractor and is a superattractor for $K = -1$.*
2. *If $|K+1| = 1$, then $z = 1$ is a parabolic fixed point.*
3. *If $|K+1| > 1$, then $z = -$ is a repulsive fixed point.*

2.3 Study of the critical points

Critical points of $S(z)$ satisfy $S'(z) = 0$, that is, $z = 0$, $z = \infty$ and

$$C_1 = \frac{2K}{K-1}; \quad (K \neq 1) \tag{2.4}$$

In case that $K = 1$ the critical points are $z = 0$ and $z = \infty$.

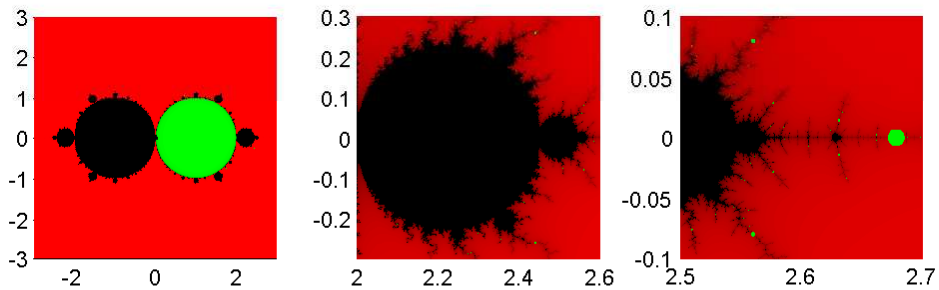

Figure 1: Parameter space associated to the critical point C_1 and zoom.

2.4 Study of parameter space

In order to find the best members of the family in terms of stability, the parameter space associated to the free critical point C_1 will be shown. It is well known that there is at least one critical point associated with each invariant Fatou component. The parameter plane is obtained by iterating the selecting critical point; each point of the parameter plane is associated with a complex value of K. Here, we using a mesh of 1000×1000 points, a maximum of 50 iterations and a tolerance of 10^{-2}. Green color in Figure 1 means that the critical point is in the basin of attraction of $z = 0$; if this is red is in the basin of attraction of $z = \infty$, whereas that black color indicates that the critical point generates iterations do not converge.

2.5 Dynamical Planes

Then, focussing the attention in the regions shown in Figure 1 it is evident that there exist members of the family with complicated behavior. In Figures 2-5 diverse dynamical planes are shown. In these dynamical planes the convergence to 0 appear in red, in green it appears the convergence to ∞, in black the zones with no convergence to the roots. These figures are similar to those presented in [9] and [11]. The degradation of colors is related to the speed of convergence.

2.6 Basins of attraction

Using $f(x) = (x-1)^m(x+1)^n$ with $m = Kn$ and real part of m and n greater than zero in (1.1) the following equation of iteration is obtained

$$x_{r+1} = x_r - \frac{K(x_r-1)(x_r+1)}{(K+1)x_r + K - 1}, \quad r = 0, 1, 2, \cdots \tag{2.5}$$

To present the basins of attraction of the fixed points $x = \pm 1$ of the function of iteration

$$G(x) = x - \frac{K(x-1)(x+1)}{(K+1)x + K - 1} \tag{2.6}$$

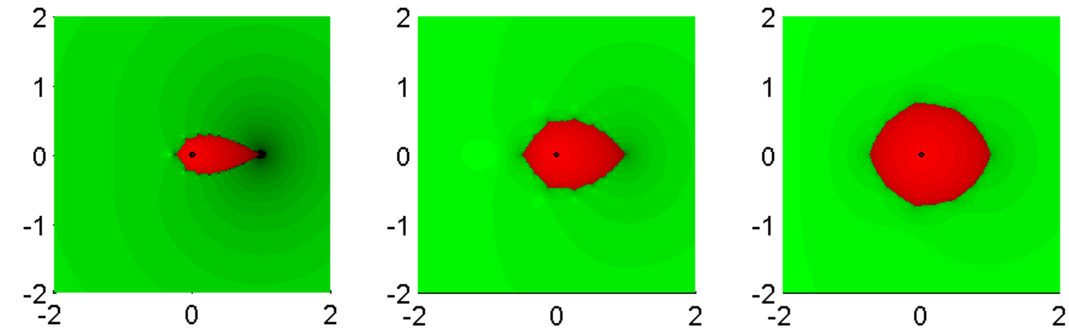

Figure 2: Dynamical planes. $K = 0.25, 0.5, 0.75$

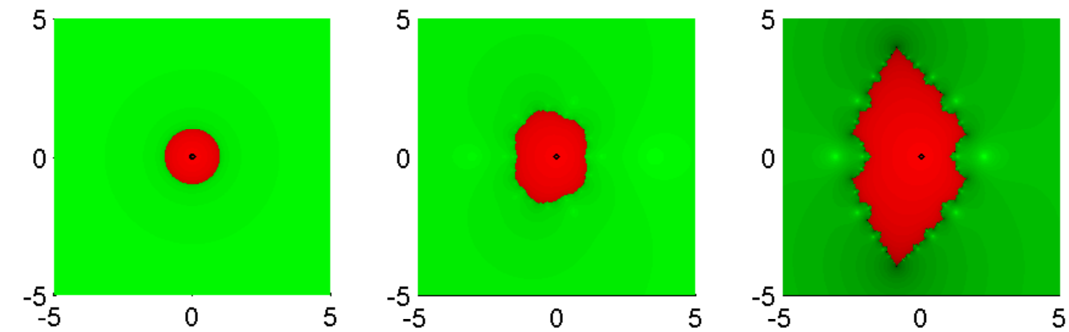

Figure 3: Dynamical planes. $K = 1, 1.4, 1.8$

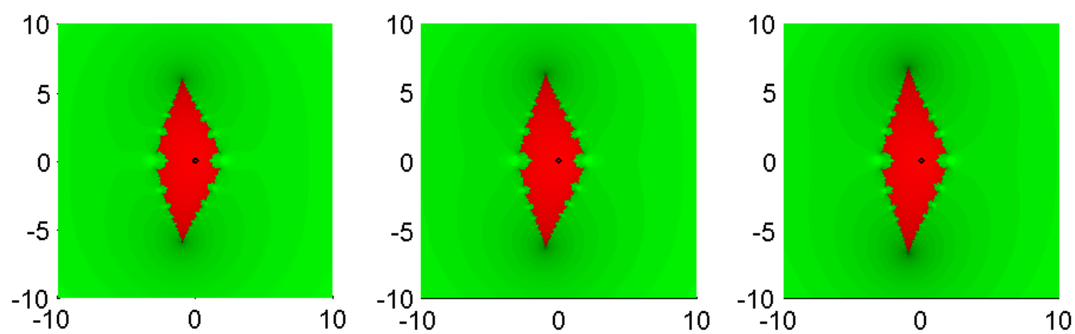

Figure 4: Dynamical planes. $K = 1.9, 1.91, 1.92$

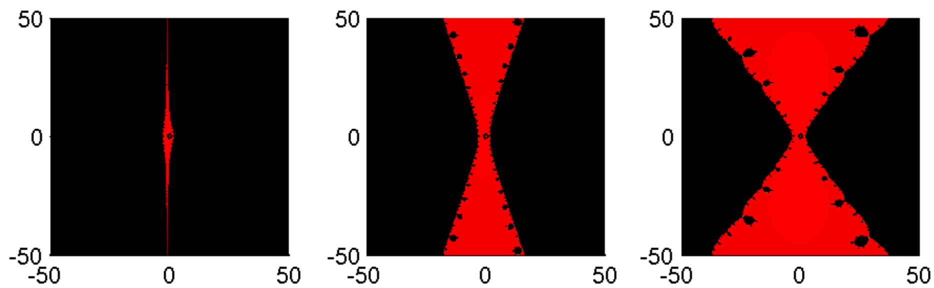

Figure 5: Dynamical planes. $K = 2, 2.125, 2.25$

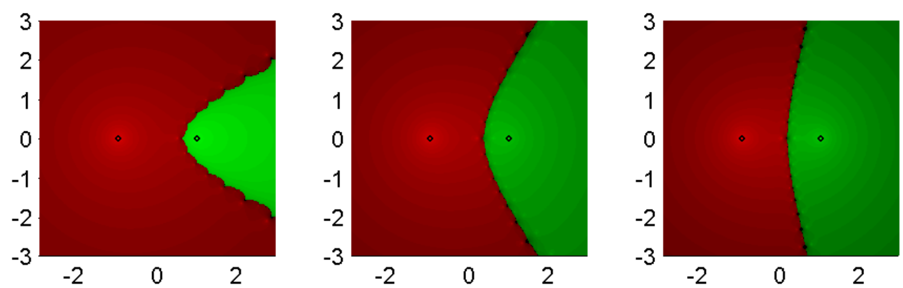

Figure 6: Basins of attraction. $f(x) = (x-1)^m(x+1)^n$, $m = Kn$, $K = 0.25, 0.5, 0.75$

associated to (2.5) we using a mesh of 1000×1000 points, a maximum of 30 iterations and a tolerance of 10^{-2} in Figures 6 and 7.

In Figures 8 and 9 we using a mesh of 250×250 points and to decrease the zones of non-convergence we use a maximum of 60 of iterations. In the Figure 10 a maximum of 20 iterations is used to observe in detail the speed of convergence.All the figures presented in this work were made by adapting the code presented in [5].

3 Final Remarks

In this paper we present a new approach to study the dynamics of Modified Newton's method. This approach is easily adapted to other methods for multiple roots. Here, the scaling theorem and the conjugation mapping for Modified Newton's method were first established, then the fixed points and critical points of the obtained rational operator were studied. We also analyzed the parameter space, selecting different values of this parameter to make the respective dynamic planes. It is clear that more studies on the dynamics of this family are necessary.

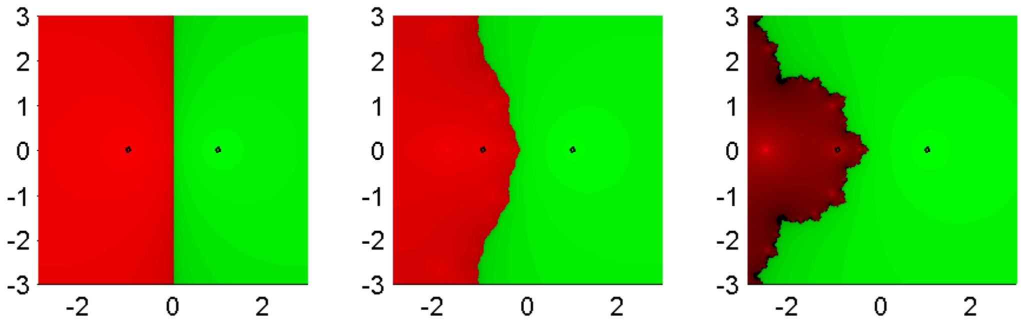

Figure 7: Basins of attraction. $f(x) = (x-1)^m(x+1)^n$, $m = Kn$, $K = 1, 1.4, 1.8$

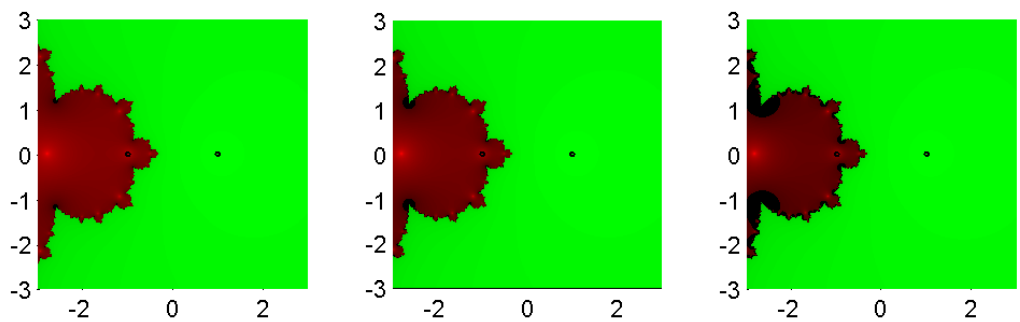

Figure 8: Basins of attraction. $f(x) = (x-1)^m(x+1)^n$, $m = Kn$, $K = 1.9, 1.91, 1.92$

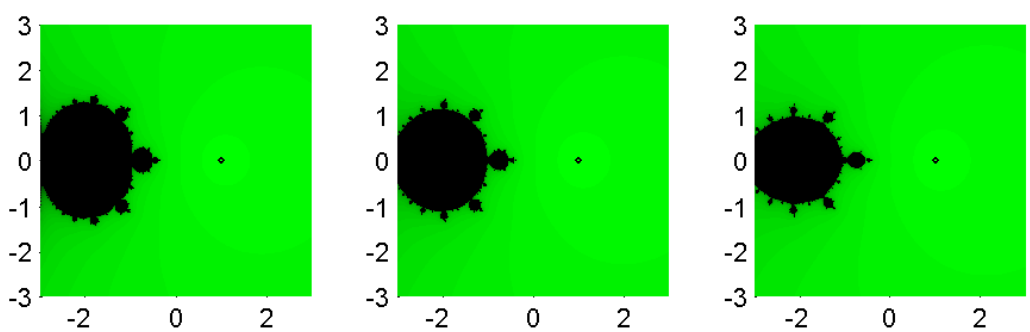

Figure 9: Basins of attraction. $f(x) = (x-1)^m(x+1)^n$, $m = Kn$, $K = 2, 2.125, 2.25$

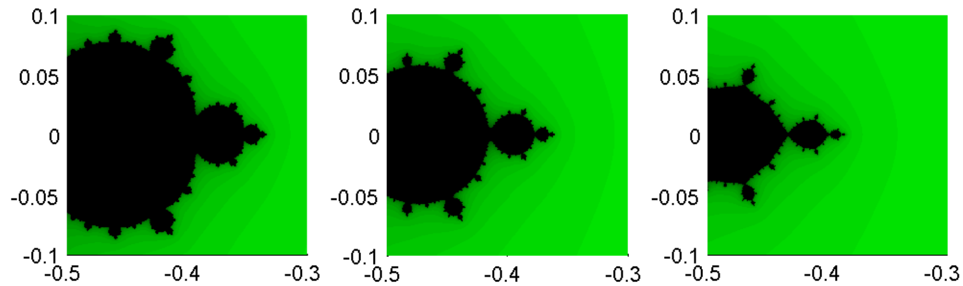

Figure 10: Zoom of basins of attraction. $f(x) = (x-1)^m(x+1)^n$, $m = Kn$, $K = 2, 2.125, 2.25$

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